CyberChallenge.IT 2022 - Test
Commented solutions

Contents

1 Question 1

1.1 Question

Alberto, Barbara, Carlo and Dario compete in a race. We know that:

- Dario finished before Alberto
- Barbara wasn't third
- There were two runners between Alberto and Carlo
- There were no more runners in the race

Who won the race?

1.2 Answers

(A) Barbara

(B) Alberto

(C) Carlo

(D) Dario

1.3 Proposed solution

The correct answer is `(C) Carlo`.

From the third statement we know that Alberto and Carlo are the first and the last, in some order, but the first statement tells us that Alberto did not win, so Carlo conquered the first position.

2 Question 2

2.1 Question

On an island there are truth-tellers and liars. You meet 2022 of them, sitting in a circle. Everyone of them says "*I am a truth-teller, or the guy sitting at my left is*".

Knowing that at least one of them is a liar, how many liars are there? (Here the "*or*" is inclusive).

2.2 Answers

(A) 2022

(B) 1011

(C) 1010

(D) 1

2.3 Proposed solution

The correct answer is `(A) 2022`.

Let us suppose that the first guy we speak to is a liar. Then, the one at his left is necessarily a liar, otherwise the first one would tell the truth. The reasoning can be iterated until we can state that each one of the 2022 is a liar.

3 Question 3

3.1 Question

You have 3 stacks of coins, with respectively 2022, 2021 and 2020 coins. Every minute, you remove 3 coins from the biggest stack, add one coin to each of the others and throw away the last coin.

After how many minutes one of the three stacks will reach 0 coins for the first time?

3.2 Answers

(A) 6059

(B) 6058

(C) 6060

(D) 6057

3.3 Proposed solution

The correct answer is `(B) 6058`.

The first three steps are the following:

$$(2022, 2021, 2020) \to (2019, 2022, 2021) \to (2020, 2019, 2022) \to (2021, 2020, 2019)$$

We notice that at the third minute each stack decreased by one with respect to the first one. We can thus state that after $2019 \cdot 3 = 6057$ minutes the three stacks will have $(3, 2, 1)$ coins. At the 6058^{th} minute the first stack will then reach the value 0.

4 Question 4

4.1 Question

Let S be a subset of $(1, 2, \dots, 2022)$. S is such that if we take x and y randomly from S, the sum $(x + y)$ will never be divisible by 5.
What is the maximum possible size of S?

4.2 Answers

(A) 809

(B) 3

(C) 407

(D) 811

4.3 Proposed solution

The correct answer is `(D) 811`.

Let us consider the classes of residues modulo 5[1]. If the sum $x + y$ is divisible by 5, we can write $x + y \equiv 0 \pmod 5$ or equivalently $x \equiv -y \pmod 5$.

There are 404 elements in each class with remainder 0, 3 or 4, while classes 1 and 2 have 405 elements. These two classes can be in S together, and they prevent the addition of elements from classes 4 and 3, respectively.

Finally, we can add to S one of the elements inside class 0, for a total of $405 + 405 + 1 = 811$ elements in S.

[1] See `https://en.wikipedia.org/wiki/Modular_arithmetic`

5 Question 5

5.1 Question

Gaspare has a 3×3 square, divided in 9 little identical squares. He paints randomly every little square black or white (each color is equally likely).

He then rotates the big square by 90° counterclockwise about its center, and every black square that now is in a position that previously was white is now painted white.

What is the probability that all the square is now white?

5.2 Answers

(A) `49/512`

(B) `7/64`

(C) `121/1024`

(D) `9/32`

5.3 Proposed solution

The correct answer is `(A) 49/512`.

The central square must be white since the beginning, otherwise it would not change its color. This happens with probability $\frac{1}{2}$.

Moreover, the little squares at the edges and the little squares at the corners form two independent groups. Each of the two groups has 16 possible configurations and 7 of them are good for the purpose of the question:

- the configuration in which every little square (among the four in the group) is already white,
- the four configurations in which one of the little squares is black and the others are white,
- the two configurations in which the four squares have alternating colours.

To sum up, the probability that the new square is white is $\frac{1}{2} \cdot \frac{7}{16} \cdot \frac{7}{16} = \frac{49}{512}$.

6 Question 6

6.1 Question

Alberto, Barbara and Carlo play a game. In this game, Alberto wins with probability $\frac{1}{2}$, and Barbara's winning probability is two times Carlo's one.

What is the probability that after 6 matches Alberto won exactly 3 times, Barbara 2 times and Carlo 1?

6.2 Answers

(A) `1/432`

(B) `5/36`

(C) `1/4`

(D) `1/6`

6.3 Proposed solution

The correct answer is `(B) 5/36`.

Let p_a, p_b, p_c be the winning probabilities of Alberto, Barbara and Carlo, respectively. Then, p_b is $(1 - p_a) \cdot \frac{2}{3} = \left(1 - \frac{1}{2}\right) \cdot \frac{2}{3} = \frac{1}{3}$, while Carlo's one is $\frac{p_b}{2} = \frac{1}{6}$.

Fixed the order of the matches won by the three people, the probability that it would happen is $\frac{1}{2^3} \cdot \frac{1}{3^2} \cdot \frac{1}{6}$. But there are $\frac{6!}{3! \cdot 2!}$ possible orderings for the winners (i.e. the number of anagrams of the string "AAABBC"[2]).

Therefore the total probability is $\frac{1}{2^3} \cdot \frac{1}{3^2} \cdot \frac{1}{6} \cdot \frac{6!}{3! \cdot 2!} = \frac{5}{36}$.

[2] See `https://en.wikipedia.org/wiki/Permutation#Permutations_of_multisets`

7 Question 7

7.1 Question

Mario walks down a street with 4 houses, colored in Red, Orange, Blue and Yellow in some order. He passes the Orange house before the Red one, and the Blue house before the Yellow one.

Knowing that there is at least one other house between the Yellow and the Blue house, how many possible orders of the four colors are there?

7.2 Answers

(A) 4

(B) 2

(C) 3

(D) 5

7.3 Proposed solution

The correct answer is `(C) 3`.

Let us denote the houses by the initial letter of their color. We know that O comes before R and B comes before Y. It is possible in the following cases: $OBRY$, $BORY$, $BOYR$, $OBYR$, $ORBY$ and $BYOR$.

But the last three are in contrast with the last information given in the question, because no house is between B and Y. This leaves us with 3 possible arrangements of the houses.

8 Question 8

8.1 Question

Marco and Giovanni both have n coins. Marco arranges his coins in groups of exactly 3 elements, while Giovanni in groups of exactly 4 of them.

Knowing that Giovanni made 4 fewer groups, what is the value of n?

8.2 Answers

(A) 48

(B) 12

(C) 24

(D) 36

8.3 Proposed solution

The correct answer is `(A) 48`.

Let $x = \dfrac{n}{3}$ be the number of groups made by Marco and $y = \dfrac{n}{4}$ the ones made by Giovanni. Then:

$$\begin{cases} x = y + 4 \\ 3x = 4y \end{cases} \qquad \begin{cases} x = y + 4 \\ 3y + 12 = 4y \end{cases} \qquad \begin{cases} x = y + 4 \\ y = 12 \end{cases} \qquad \begin{cases} x = 16 \\ y = 12 \end{cases}$$

We can now recover the value of $n = 3x = 4y = 48$.

9 Question 9

9.1 Question

Barbara draws 3 possibly overlapping triangles on an infinite sheet of paper. What is the maximum possible number of regions, including the exterior region, that the paper can be divided into by the sides of the triangles?

9.2 Answers

(A) 12

(B) 19

(C) 22

(D) 20

9.3 Proposed solution

The correct answer is `(D) 20`.

Let us consider the resulting figure as if it was a planar graph and exploit the Euler's formula[3] to answer the question.

It is easy to see that each edge of the triangles can cross at most other four, so the number of intersection points between them is $\frac{4 \cdot 9}{2} = 18$, since we have 9 edges and we are counting the points twice. Later, we will need to add the 9 vertex of the triangles to make the formula work. Moreover, by the same argument as before, the maximum number of segments (that are the edges of the planar graph) is $5 \cdot 9 = 45$.

We are now ready to compute the number of regions. The formula is

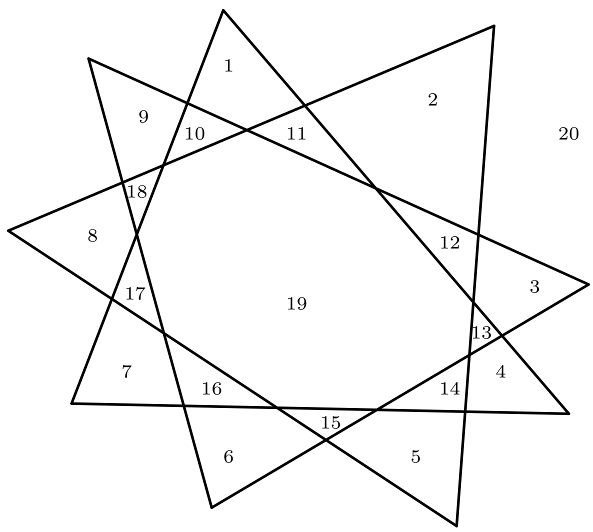
$$v - e + f = 2$$

where v is the number of vertex, e the number of edges and f the number of faces. We have then

$$(18 + 9) - 45 + f = 2$$

$$f = 2 + 45 - 27 = 20$$

The number of region is therefore 20. An example figure is reported below.



[3]See `https://en.wikipedia.org/wiki/Planar_graph#Euler's_formula`

10 Question 10

10.1 Question

You have four fair 6-sided dice, numbered from 1 to 6. You roll all of them.

What is the probability that the product of the results is a prime number?

10.2 Answers

(A) `1/1296`

(B) `1/2`

(C) `1/162`

(D) `1/108`

10.3 Proposed solution

The correct answer is `(D) 1/108`.

In order to be prime, the product of the numbers on the dice must be formed by three 1 and one among $2, 3, 5$. Moreover, these four numbers can appear in four different orders, so we have $4 \cdot 3 = 12$ favourable outcomes, while the total ones are obviously 6^4.

Thus, the probability is $\frac{12}{6^4} = \frac{1}{108}$.

11 Question 11

11.1 Question

Given the following function:

```
function f(a, b) {
        if(b == 0)
                return a
        if(b > 0)
                return 3 + f(a, b - 1)
        return f(a, b + 1) - 3
}
```

What does `f(a,b)` calculate?

11.2 Answers

(A) `a-3b`

(B) `a+3b`

(C) `3ab`

(D) `3a-b`

11.3 Proposed solution

The correct answer is `(B) a+3b`.

The function `f` calls itself recursively until `b` is zero, and at each call it subtracts (resp. add) 1 from `b` and add (resp. subtract) 3 to the final result. This means that the result to be added to `a` is `3b` (either if `b` is positive or negative), so the function compute the value `a+3b`.

12 Question 12

12.1 Question

Consider the following function:

```
function s(v, p, n) {
        int u[n]
        for(i=0; i<n; i++) {
                u[p[i]] = v[i]
        }
        return u
}

function f(n) {
        v = [0,1,2,3,4,5,6]
        p = [5,1,4,0,6,2,3]
        for(i=0; i&lt;n; i++)
                v = s(v, p, 7)
        return v
}
```

What is the results of f(65536)?

12.2 Answers

(A) `[2, 1, 6, 5, 3, 4, 0]`

(B) `[0, 1, 2, 3, 4, 5, 6]`

(C) `[3, 1, 5, 6, 2, 0, 4]`

(D) `[5, 1, 4, 0, 6, 2, 3]`

12.3 Proposed solution

The correct answer is `(A) [2, 1, 6, 5, 3, 4, 0]`.

The code of function `s` is returning a permutation of vector `v` described by vector `p`, while the function `f` is iterating that permutation.

We can compute the first iterations looking for the initial vector: that would mean that we found the maximum length of the permutation cycle[4].

```
f(1)  = [3, 1, 5, 6, 2, 0, 4]
f(2)  = [6, 1, 0, 4, 5, 3, 2]
f(3)  = [4, 1, 3, 2, 0, 6, 5]
f(4)  = [2, 1, 6, 5, 3, 4, 0]
f(5)  = [5, 1, 4, 0, 6, 2, 3]
f(6)  = [0, 1, 2, 3, 4, 5, 6]
```

So the permutation is repeating itself every 6 iterations. At this point, we can write
`f(65536) = f(10922*6+4) = f(4) = [2, 1, 6, 5, 3, 4, 0]`.

[4]See `https://en.wikipedia.org/wiki/Cyclic_permutation`, that also suggests a more efficient method to compute the cycle length

13 Question 13

13.1 Question

Consider the following function:

```
function f(n) {
        if(n < 4)
                return 1
        return 2*f(n-1) - 3*f(n-2) + 1*f(n-1)
}
```

What is the value of `f(75) - f(74)`?

13.2 Answers

(A) 50031545098999707

(B) -150094635296999121

(C) 0

(D) 3

13.3 Proposed solution

The correct answer is `(C) 0`.

Let use notice that

$$f(0) = f(1) = f(2) = 1$$

.

Let us rewrite the function f as $f(n) = 3f(n-1) - 3f(n-2)$ if $n \geq 4$, thus $f(4) = 3f(3) - 3f(2) = 3 - 3 = 0$. Moreover we have, for $n \geq 4$:

$$\begin{aligned}
f(n+6) &= 3f(n+5) - 3f(n+4) \\
&= 9f(n+4) - 9f(n+3) - 3f(n+4) = 6f(n+4) - 9f(n+3) \\
&= 18f(n+3) - 18f(n+2) - 9f(n+3) = 9f(n+3) - 18f(n+2) \\
&= 27f(n+2) - 27f(n+1) - 18f(n+2) = 9f(n+2) - 27f(n+1) \\
&= 27f(n+1) - 27f(n) - 27f(n+1) = -27f(n)
\end{aligned}$$

This means that, starting from 4, if $f(n) = 0$ then also $f(n+6) = 0$. Let us notice that $f(76) = f(4+12 \cdot 6) = 0$, but also $f(76) = 3f(75) - 3f(74)$, so $f(75) - f(74) = 0$.

14 Question 14

14.1 Question

Consider the expression x^{30}, which is the minimum number of multiplications necessary to calculate its value?

14.2 Answers

(A) 6

(B) 5

(C) 7

(D) 29

14.3 Proposed solution

The correct answer is `(A) 6`.

The minimum number of multiplications necessary to exponentiate a number is 6:

- compute x^2 as $x \cdot x$
- compute x^4 as $x^2 \cdot x^2$
- compute x^5 as $x^4 \cdot x$
- compute x^{10} as $x^5 \cdot x^5$
- compute x^{15} as $x^{10} \cdot x^5$
- compute x^{30} as $x^{15} \cdot x^{15}$

On the other side, five multiplications are not enough since at the fourth one we need to have at least x^{15}. But x^{15} requires five multiplications, while x^{16} is the maximum that we can construct in four steps; still, we then would need to multiply by x^{14}, that we do not have available, to conclude.

15 Question 15

15.1 Question

Consider the following operation:

```
((x ^ 31) & 15)
```

What is the lowest number should I put in place of x to get 2?

15.2 Answers

(A) 2

(B) 29

(C) 15

(D) 13

15.3 Proposed solution

The correct answer is `(D) 13`.

Let us consider the two operations separately. The XOR with 31 is flipping the five least significant bits and preserving the others, while the AND with 15 is keeping the four least significant bits and discarding the others.

This means that the four least significant bits of `x` must form the number 13, while the fifth one can be either 0 or 1. But setting it to 1 would increase `x` to 29, while we are looking for the lowest value, that is indeed 13.