CyberChallenge.IT 2021 - Test
Commented solutions

Contents

1 Question 1

1.1 Question

Alice writes down 5 consecutive positive integer, then it takes 4 of them and sum them up, obtaining 217.

What is the sum of the 5 initial integers?

1.2 Answers

(A) 271

(B) 272

(C) 270

(D) 1024

1.3 Proposed solution

The correct answer is `(C) 270`.

Let us call the five numbers $x-2$, $x-1$, x, $x+1$, $x+2$ for some integer x. Then their sum is $5x$ (note that, given the answers, this is enough to answer, since only one possibility is a multiple of 5, but it is possible to find the value without looking at the proposed values).

The sum of four of the five numbers is $4x \pm \varepsilon = 217$ for some integer $0 \leq \varepsilon \leq 2$. We can then write $4x = 217 \pm \varepsilon$ and notice that the right hand side must be a multiple of 4, which implies that $\varepsilon = -1$, given the constraints on this variable.

Therefore,

$$\begin{aligned} 4x &= 216 \\ x &= 54 \\ 5x &= 270. \end{aligned}$$

2 Question 2

2.1 Question

Bob has 120 cards. He decides to give *all of them* to his 3 children of 4, 7, 9 years.

He gives each child a number that is some positive integer constant c (that is the same for every child) multiplied by his age.

How many cards will receive the oldest child?

2.2 Answers

(A) 36

(B) 54

(C) 45

(D) 63

2.3 Proposed solution

The correct answer is `(B) 54`.

Given the constraints in the question, we can write:

$$\begin{aligned} 4c + 7c + 9c &= 120 \\ 20c &= 120 \\ c &= 6. \end{aligned}$$

The oldest child will then receive $c \cdot 9 = 6 \cdot 9 = 54$ cards.

3 Question 3

3.1 Question

What is the smallest positive integer that can be written as the product of 6 different integers (except 0)?

3.2 Answers

(A) 36

(B) 720

(C) 1440

(D) 48

3.3 Proposed solution

The correct answer is `(D) 48`.

In order to keep the product of the six integers as small as possbile, we need to keep (the absolute value of) the six integers as small as possible. As a consequence, we are going to choose two negative and four positive integers or vice-versa (it would be better to divide the integers into two groups of three, but then the product would be negative).

Our choice would then be $1, -1, 2, -2, 3, 4$ and their product is $(-2) \cdot (-1) \cdot 1 \cdot 2 \cdot 3 \cdot 4 = 48$.

4 Question 4

4.1 Question

Three people are in a room with 4 walls and 4 doors.

The first one says: “No doors are on the north wall”,

the second one: “All the doors are on the south wall”,

the third one: “On every wall there is at most one door”.

Knowing that every person may have lied, what can we say?

4.2 Answers

(A) The third one is telling the truth

(B) Everyone is lying

(C) At least one of them is lying

(D) Nothing

4.3 Proposed solution

The correct answer is `(C) At least one of them is lying`.

The sentence from the third person implies that each wall has exactly a door, since there are an equal number of walls and doors. But it is also in contrast with what is said by the second person, so answer `(C)` is the valid one.

5 Question 5

5.1 Question

On a island there are 3 kind of people:

1. *truth-tellers*, that always say the truth,
2. *liars*, that always lie,
3. *impostors*, that can both tell the truth or lie, but if they are not the first to speak they lie if and only if the one that spoke immediatly before them told the truth.

Alice, Bob and Carol tell the following sentences:

- Alice: There is at least a truth-teller among us
- Bob: I am a liar
- Carol: There are no impostors among us

Which of the following is true?

5.2 Answers

(A) Among the three there is exactly one truth-teller, one liar and one impostor

(B) There are at least two truth-tellers

(C) There are at least two liars

(D) For at least two of them we don't have enough information to determine their type

5.3 Proposed solution

The correct answer is `(A) Among the three there is exactly one truth-teller, one liar and one impostor`.

Bob is necessarily a lying impostor, since otherwise we would have a contraddiction. Thus, Carol is a liar (since Bob is an impostor, she does not tell the truth and she cannot be another lying impostor) and Alice is telling the truth (since Bob is a lying impostor) and thus she must be a truth-teller.

Therefore, there is exactly one representative for each class.

6 Question 6

6.1 Question

Let x and y be two positive integers strictly greater than 1. You know that the product $x \cdot y$ divides 100 (that is, the fraction $\frac{100}{x \cdot y}$ is an integer).

What can you say about x and y?

6.2 Answers

(A) At least one of them is even

(B) If x is even then y is a multiple of 5

(C) There are at least 2 correct sentences in the other 3 answers

(D) x is even or multiple of 5 (or both)

6.3 Proposed solution

The correct answer is `(D) x is even or multiple of 5 (or both)`.

Since x is strictly greater than 1 and it should divide $100 = 2^2 \cdot 5^2$, it must have at least one of the factors of 100 in its factorization, and the same holds for y. This means that each of the two variables must be even or multiple of five, or both, since 2 and 5 are the only factors of 100.

Instead `(A)` and `(B)` are incorrect since $(x, y) = (5, 5)$ and $(x, y) = (2, 2)$ are counterexamples.

7 Question 7

7.1 Question

Carol has a bunch of 8 indistinguishable keys on a ring. She wants to color them using n colors in a way such that she can recognize exactly each key. What is the smallest possible value of n?

7.2 Answers

(A) 3

(B) 4

(C) 2

(D) 5

7.3 Proposed solution

The correct answer is `(C) 2`.

Carol needs a way to identify the orientation of the keyring and a “starting point”, since the ring is circular. Let us show that two colours are sufficient.

Let us consider the sequence of colours $ABAABBBB$. It allows us to have a starting point: we just need to start from the single A. Moreover, this way we are able to find out if the ring is in the correct orientation, since after our starting point we should find the sequence BAA and not BBB (as would happen if the keyring is flipped).

8 Question 8

8.1 Question

In a football championship 20 teams play against each other team exactly one time. What is the maximum number of teams that can end up with 16 wins or more?

8.2 Answers

(A) 6

(B) 7

(C) 8

(D) 5

8.3 Proposed solution

The correct answer is `(B) 7`.

Let k be the answer. Then, in the optimal configuration each of the k teams wins against the other $20 - k$ and wins against half of the remaining $k - 1$ adversaries. Since the number of wins is at least 16, we can write:

$$\begin{aligned} 16 &\leq (20 - k) + \frac{k-1}{2} \\ 32 &\leq 40 - 2k + k - 1 \\ k &\leq 40 - 1 - 32 = 7. \end{aligned}$$

So, the maximum possible value for k is 7.

9 Question 9

9.1 Question

Let Y be a year in the 21st century. Knowing that the last day of year Y is a Tuesday, the first day of the year $Y + 2$ is a Friday, on what day will be the last Sunday of January in the year $Y + 3$?

9.2 Answers

(A) 30

(B) 28

(C) 29

(D) 31

9.3 Proposed solution

The correct answer is `(A) 30`.

The last day of year Y is Tueasday, so the first one of year $Y + 1$ is Wednesday. Then, since the first day of year $Y + 2$ is Friday, we can conclude that year $Y + 1$ is a leap year and as a consequence year $Y + 2$ is not.

Thus, the first day of year $Y + 3$ is Saturday and the days of year $Y + 3$ that are a Sunday are $2, 9, 16, 23, 30$.

10 Question 10

10.1 Question

100 people are in a row, each one being a truth-teller, that always tells the truth, or a liar, that always lies.

Each of them says: "Among the people not in front of me, including myself, more than one half are truth-tellers".

How many possible configurations of truth-teller/liars are possible?

10.2 Answers

(A) 100

(B) 102

(C) 200

(D) 101

10.3 Proposed solution

The correct answer is `(D) 101`.

If the last one (the one that has nobody behind him) lies, then necessarily everyone in the row lies. Otherwise, if that one is a truth-teller, we analyze the second to last: if he is again a truth-teller, than the entire row is made up of truth-tellers, otherwise we can "detach" the last two people and see what happens for the row with 98 people.

We can count the number of possible configurations finding a recursive law: let x_n be the number of configurations with n people in the row. It holds that $x_2 = 3$, in fact we can have TT, TL, LL, where T stands for a truth-teller and L for a liar. Moreover, thanks to the previous rasoning, we can write $x_{n+2} = x_n + 2$ that implies $x_{100} = 101$.

11 Question 11

11.1 Question

Consider the following function:

```
function s(v, p, n) {
        int u[n]
        for(i=0; i < n; i++) {
                u[p[i]] = v[i]
        }
        return u
}

function f(n) {
        v = [0,1,2,3,4,5,6]
        p = [2,3,4,5,6,1,0]
        for(i=0; i < n; i++)
                v = s(v, p, 7)
        return v
}
```

What is the result of `f(1000000000000)`?

11.2 Answers

(A) `[2, 3, 4, 5, 6, 1, 0]`

(B) `[2, 1, 4, 3, 6, 5, 0]`

(C) `[0, 5, 2, 1, 4, 3, 6]`

(D) `[6, 3, 0, 5, 2, 1, 4]`

11.3 Proposed solution

The correct answer is `(C) [0, 5, 2, 1, 4, 3, 6]`.

The code of function `s` is returning a permutation of vector `v` described by vector `p`, while the function `f` is iterating that permutation.

We can compute the first iterations looking for the initial vector: that would mean that we found the maximum length of the permutation cycle[1].

```
f(1)  = [6, 5, 0, 1, 2, 3, 4]
f(2)  = [4, 3, 6, 5, 0, 1, 2]
f(3)  = [2, 1, 4, 3, 6, 5, 0]
f(4)  = [0, 5, 2, 1, 4, 3, 6]
f(5)  = [6, 3, 0, 5, 2, 1, 4]
f(6)  = [4, 1, 6, 3, 0, 5, 2]
f(7)  = [2, 5, 4, 1, 6, 3, 0]
f(8)  = [0, 3, 2, 5, 4, 1, 6]
f(9)  = [6, 1, 0, 3, 2, 5, 4]
f(10) = [4, 5, 6, 1, 0, 3, 2]
f(11) = [2, 3, 4, 5, 6, 1, 0]
f(12) = [0, 1, 2, 3, 4, 5, 6]
```

So the permutation is repeating itself every 12 iterations. At this point, we can write
`f(1000000000000) = f(83333333333*12+4) = f(4) = [0, 5, 2, 1, 4, 3, 6]`.

[1]See `https://en.wikipedia.org/wiki/Cyclic_permutation`, that also suggests a more efficient method to compute the cycle length

12 Question 12

12.1 Question

Consider the following function:

```
function f(n) {
        if(n < 3) {
                return 1
        }
        return 2*f(n-1)-3*f(n-2)+2*f(n-3)
}
```

What is the results of `f(96)`?

12.2 Answers

(A) 1

(B) 0

(C) 2

(D) -1

12.3 Proposed solution

The correct answer is `(A) 1`.

Let use notice that

$$f(0) = f(1) = f(2) = 1$$

and that, if

$$f(n-1) = f(n-2) = f(n-3) = f$$

then

$$f(n) = 2 \cdot f(n-1) - 3 \cdot f(n-2) + 2 \cdot f(n-3) = 2f - 3f + 2f = f$$

This is sufficient to conclude that the function `f` will always return 1 regardless of the value of `n`.

13 Question 13

13.1 Question

Consider the following function:

```
function f(a,b,c) {
        if(a < 10 $x b > 50)
                return true
        if(b > 45 $y c < 70)
                return true
        if(a > 3 $z c < 40)
                return true
        return false
}
```

What do you have to put in place of `$x`, `$y` and `$z` to makes `f(2, 0, 70)`, `f(3, 20, 70)` and `f(3, 40, 80)` true?

13.2 Answers

(A) `and, and, and`

(B) `or, and, or`

(C) `and, or, and`

(D) `and, and, or`

13.3 Proposed solution

The correct answer is `(B) or, and, or`.

The second and the third conditions are useless for our purposes, since we have no triplet that has to be evaluated and meet at least one of those conditions. We have to take car of the first only.

Again, we have no triplet that meets the condition `b > 50`, so an `and` would not work. Among the proposed answer, only one has an `or` for the first `if`, so that is the correct one.

14 Question 14

14.1 Question

Consider the expression x^{15}, which is the minimum number of multiplications necessary to calculate its value?

14.2 Answers

(A) 6

(B) 5

(C) 14

(D) 4

14.3 Proposed solution

The correct answer is `(B) 5`.

The minimum number of multiplication necessary to exponentiate a number is 5:

- compute x^2 as $x \cdot x$
- compute x^4 as $x^2 \cdot x^2$
- compute x^5 as $x^4 \cdot x$
- compute x^{10} as $x^5 \cdot x^5$
- compute x^{15} as $x^{10} \cdot x^5$

On the other side, four multiplications are not enough since at the third one we need to have at least x^8, which is also the maximum that we can construct in three steps; still, we then would need to multiply by x^7, that we do not have available, to conclude.

15 Question 15

15.1 Question

Given the following functions:

```
function x(n) {
        if(n <= 0) return 0
        return 1+y(n-2)
}

function y(n) {
        if(n <= 0) return 0
        return 1+x(n-3)
}
```

How many times the functions are called with x(30000)?

15.2 Answers

(A) 12001

(B) 15001

(C) 12000

(D) 15000

15.3 Proposed solution

The correct answer is `(A) 12001`.

Considering one call to `x` and one to `y`, the argument `n` is decreased by 5, while the (partial) return value is increased by 2. This means that with 6000 calls to `x` (and 6000 to `y`), the program is going to call `x(0)`. In total we called the functions $6000 + 6000 + 1 = 12001$ times.